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*Published in:*  
 Proceedings of the 2002 American Control Conference

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*Document Version*  
 Publisher's PDF, also known as Version of record

*Publication date:*  
 2002

[Link to publication in University of Groningen/UMCG research database](#)

*Citation for published version (APA):*

Clemente-Gallardo, J., Jeltsema, D., & Scherpen, J. M. A. (2002). Algebroids and Charge Conservation. In *Proceedings of the 2002 American Control Conference* (Vol. 5, pp. 3552-3557). University of Groningen, Research Institute of Technology and Management.

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# Algebroids and Charge Conservation

Jesús Clemente-Gallardo, Dimitri Jeltsema and Jacquélien M.A. Scherpen<sup>1</sup>

Systems and Control Engineering Laboratory,  
Faculty of Information Technology and Systems,  
Delft University of Technology  
P.O. Box 5031, 2600 GA Delft, The Netherlands.  
E-mail: j.m.a.scherpen@its.tudelft.nl

**Abstract** – This paper presents two new choices of regular Lagrangians for the dynamical description of non-switched LC networks, and proves that the solutions are equivalent to each other and to the results obtained in the literature. The first approach uses the integrated version of Kirchhoff's current law (topological constraints), which is just the condition of charge conservation, to define a regular Lagrangian description by using only the inductances of the system. The second approach is based on the formulation of a Lie algebroid which defines the constraints of the system in a different way. Both approaches are shown to be in a one-to-one relation and they both provide equivalent Hamiltonian formalisms.

## 1 Introduction

In the last years, an evident interest for the Lagrangian and Hamiltonian description of electrical circuits has arisen in the literature [1, 6, 7, 8, 11, 12]. A recent Lagrangian description [11, 12] leads to a successful picture of RLC circuit dynamics and provides a step-by-step construction for the description of the components, the definition of the Lagrangian, and the corresponding Euler-Lagrange dynamics. Kirchhoff's current law defines a set of holonomic constraints for the corresponding Lagrangian system while the corresponding voltage law defines the Euler-Lagrange equations for the system. Regarding the Hamiltonian description of the dynamics of electrical circuits, a recent and successful approach is based on the concept of Dirac structures and Port-Controlled Hamiltonian systems [6, 14]. This approach also provides a suitable description of the dynamics of the system.

It seems quite natural to compare both approaches and to try to relate the solutions of both methods for electrical circuits. Since dissipative elements and sources can be viewed as external elements, we only consider electrical LC circuits here. The formulation of both

frameworks is done in  $\mathbb{R}^n$  spaces and hence the canonical procedure would suggest to use the Legendre transform to go from dynamics given by the Lagrangian formalism into dynamics given by the Hamiltonian formalism, and vice versa. The problem in this case is that the Lagrangian formalism proposed in [11, 12] yields a singular Lagrangian description, which makes the Legendre transform ill-defined and thus no straightforward Hamiltonian formulation can be related. In [2] we complement the original Lagrangian picture proposed in [11, 12] with a procedure that transforms the singular Lagrangian system into a regular Lagrangian system. Then the Lagrangian system can be related with a Hamiltonian system by using a well defined Legendre transform. The main new ingredient of the approach is the use of Lie algebroids in the description. A Lie algebroid is a geometrical object which generalizes the concept of tangent bundles (which is the natural framework of usual Lagrangian mechanics) such that a Lagrangian formulation on them is still possible [5, 15]. Essentially, we just need one of the simplest examples of the Lie algebroid, namely an integrable subbundle of a tangent bundle, which in the case of electrical LC circuits is a vector space. Nonetheless, the choice of coordinates in the algebroid done in [2] leads to a description of the solutions of Euler-Lagrange equations which does not correspond to the physical variables (it composes non trivially the currents of the inductances with the currents of the capacitors). Therefore, a transformation leading to the physical space (denoted by  $\mathcal{F}$  in [2]) is required.

The present paper presents two alternative choices of regular Lagrangians for the descriptions of LC networks, and proves that the solutions are equivalent one to another and to the results obtained in the above-mentioned references. The first approach uses the integrated version of the constraints, which is just the condition of charge conservation, to define a regular Lagrangian description by using only the inductances of the system. The second approach introduces another Lie algebroid formulation which defines the constraints of the system in a different way which leads to

<sup>1</sup>Corresponding author

much simpler results, and does not require the introduction of the space  $\mathcal{F}$ . If the LC network does not contain switches (as we assume throughout the paper) these two approaches are equivalent. Later we proceed with the study of the Hamiltonian formalisms associated to these regular Lagrangian systems. We prove that both approaches lead also to equivalent results and we find the equivalence between them and the Hamiltonian formalism based on the Port-Controlled Hamiltonian (PCH) framework [7] and the approach presented in [1].

The structure of the paper is as follows: Section 2 introduces very briefly the Lagrangian formulation presented in [11, 12]. Section 3 provides the main contribution of the paper: under the assumption that there are no switches on the network, the charge conservation at any point of the system determines a Lie algebroid which happens to be much more simple than the one presented in [2] and, at the same time, it allows us to describe the dynamics of the system by using only inductances or only capacitors. In section 4 the method is extended to Hamiltonian systems. Finally, Section 5 contains an example to illustrate our results.

## 2 The constraint Lagrangian formulation

In this section we present the Lagrangian formalism as introduced in [11, 12] for (switching) electrical networks. The basic construction is as follows. The system is defined on  $\text{TR}^n \sim \mathbb{R}^{2n}$ , where  $n_L$  is the number of inductances of the network,  $n_C$  the number of capacitors and  $n = n_L + n_C$ . Each element is represented by two variables, a charge (generalized position)  $q_L$  or  $q_C$  (for inductances and capacitors, respectively) and a current (generalized velocity)  $\dot{q}_L$  and  $\dot{q}_C$ . Not all generalized velocities are allowed, Kirchhoff current law is used to define a set of holonomic constraints in such a way that the dynamics is thus not defined on the whole space  $\mathbb{R}^{2n}$ , but in the subspace  $\mathcal{A} \subset \text{TR}^n$  defined by the velocities which satisfy the set of constraints. The dynamics is defined by the Lagrangian

$$L(q, \dot{q}) = \sum_{i=1}^{n_L} \frac{1}{2} L_i \dot{q}_{L_i}^2 - \sum_{j=1}^{n_C} \frac{1}{2C_j} q_{C_j}^2. \quad (1)$$

It can be seen [2, 11] that this Lagrangian is singular (in mechanical terms, its mass matrix is degenerate).

The corresponding Euler-Lagrange equations provide the suitable dynamical description, by including voltage sources as external forces and, eventually, Rayleigh dissipation terms to include resistive elements. The

Euler-Lagrange equations are given by

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}}(q, \dot{q}) \right) - \frac{\partial L}{\partial q}(q, \dot{q}) = \mathcal{V} + A^T(q)\lambda - \frac{\partial \mathcal{D}}{\partial \dot{q}}(\dot{q}) \quad (2)$$

$$A(q)\dot{q} = 0, \quad (3)$$

where  $\mathcal{V} \in \mathbb{R}^n$  denotes the external voltage sources,  $A^T(q)\lambda$  the constraint forces, with  $A(q)$  a constant  $k \times n$  matrix and  $\lambda \in \mathbb{R}^k$ , and the scalar function  $\mathcal{D}(\dot{q})$  the Rayleigh dissipation function. Additional current sources can be included in the constraint equation (3). In the above formulation we can recognize the Kirchhoff laws for the circuit, namely the dynamical equation corresponds to the voltage law and the constraint equation to the current law. In order to be able to solve the implicit system we must assume that the number of independent constraints,  $k$ , is greater or equal than the number of capacitors of the system<sup>1</sup>. For the sake of simplicity, we will assume that they are equal, i.e.,  $k = n_C$ . Without loss of generality, in the remaining of the paper only the formulation of autonomous LC circuits is considered. The external sources and dissipative elements can be viewed as external ports added to the system [7, 13], and can therefore be included afterwards.

## 3 The role of charge conservation

The definition of the admissible velocities is done many times (for instance in [2]) in the kernel representation for the constraints. If we split the set of velocities in inductance generalized velocities and capacitor generalized velocities, and consequently we write  $A(q)\dot{q} = 0 \Leftrightarrow A_L \dot{q}_L + A_C \dot{q}_C = 0$ . We can also write Kirchhoff's current law as:

$$\dot{q}_C = \hat{\Psi} \dot{q}_L, \quad (4)$$

where  $\hat{\Psi} = -A_C^\dagger A_L$  and  $A_C^\dagger$  is a left pseudo-inverse for the matrix  $A_C$  (i.e.  $A_C^\dagger A_C = \text{Id}_{n_C}$ ). The matrix  $\hat{\Psi}$  coincides with the matrix  $B_{LC}$  introduced in [1] and in [7]. This fact will be very important in the next sections. The introduction of Kirchhoff's current law can be replaced by the relation on which it is based, namely the conservation of the charge on a given node of the network. It is important to notice that we use the variable  $q_L$  to represent the electrical variable representing the electric charge of the inductor, though it has not a completely clear physical meaning. Other approaches [11, 12] have used it in the same way before. In any case, it is used as a mathematical concept which is not necessary in the analysis of the solutions of the dynamical equations. Hence, let us consider the set of linearly independent charge conservation laws on the nodes of

<sup>1</sup>This condition naturally follows from the assumption that the circuit does not contain any capacitor-only loops, inductor-only cutsets or controlled switches.

the network and, without loss of generality, let us write them in the form:

$$q_C = \Psi q_L, \quad (5)$$

where  $\Psi$  represents a  $n_C \times n_L$  matrix. This matrix can be considered to represent an application which maps the subspace defined by the inductances onto the subspace defined by the capacitors in the  $\mathbb{R}^n$  ( $n = n_L + n_C$ ) space of the charges of the network elements. If we assume that the network does not contain any switches, this matrix coincide with the expression of  $\hat{\Psi}$  (and hence with the matrix  $B_{LC}$  defined in [1] and in [7]), i.e.  $\hat{\Psi} = \Psi = B_{LC}$ . Since these subspaces are going to be very useful in the following, let us denote by  $\mathcal{L}$  the subspace of charges on the inductances, and by  $\mathcal{C}$  that of charges on the capacitors. With this notation we can write  $\Psi : \mathcal{L} \rightarrow \mathcal{C}$ . Analogously, we can consider the mapping in the other direction, by rewriting charge conservation as  $q_L = \xi q_C$  where  $\xi : \mathcal{C} \rightarrow \mathcal{L}$  is a pseudo-inverse mapping of  $\Psi$ . Let us recall that the approach followed in [11, 12] defines the Lagrangian of the system as (1). By using the mapping  $\Psi$  we can rewrite the Lagrangian above only in terms of  $q_L$  and  $\dot{q}_L$ , i.e. as a function of the tangent bundle  $T\mathcal{L}$ . We may then define a function  $L^{T\mathcal{L}}(q_L, \dot{q}_L) = L(\dot{q}_L, \Psi q_L)$  as

$$L^{T\mathcal{L}}(q_L, \dot{q}_L) = \frac{1}{2} \sum_{i=1}^{n_L} L_i \dot{q}_L^2 - \sum_{i=1}^{n_C} \frac{1}{C_i} (q_L^T \Psi^T)_i (\Psi q_L)_i. \quad (6)$$

that leads to the Euler-Lagrange equations:

$$\frac{d\dot{q}_L}{dt} = -L^{-1} \Psi^T C^{-1} \Psi q_L, \quad (7)$$

where  $L = \text{diag}(L_1, \dots, L_{n_L})$  and  $C = \text{diag}(C_1, \dots, C_{n_C})$ . Of course, we must combine this result with the equations of the conservation of the charge (5). Hence, we have defined a Lagrangian system on the space  $T\mathcal{L}$ , which turns out to be regular (the singularities coming from the absence of the capacitor generalized velocities have been removed in the process). We know from previous work (see [2]) that we can also provide a regular Lagrangian description in terms of Lie algebroids. It is quite interesting to try to find out if both treatments are related. In order to do that, we are going to introduce another formulation for the algebroid which will indeed show an evident relation between the dynamics.

It is well known that the solutions of the Euler-Lagrange equations (2) and (3) is restricted to the subbundle  $\mathcal{B}$  of the tangent bundle  $T\mathbb{R}^n$  defined by the admissible velocities. This subbundle is called *integrable* because the commutator of any two admissible velocities is also admissible; hence they define a Lie subalgebra of the Lie algebra of vector fields of the base manifold. It is also well known (see [15]) that this system is a Lie algebroid. It is a fairly simple example of

such systems, because the bundles are trivial and the base is a vector space. We say thus that the set defined by all the charges  $\mathcal{L} \times \mathcal{C}$  and the space  $F$  defined by those velocities which satisfy the constraints:

$$F = \{v \in T_q \mathbb{R}^n | A(q)v = 0\} \quad (8)$$

is a Lie algebroid included in  $T\mathbb{R}^n$ . In [2] a choice for a set of coordinates for the space  $F$  is presented. In the present paper we are going to introduce a new choice, namely we are going to consider  $F$  spanned by the rows of the matrix  $(\text{Id}_{n_L} - \Psi^T)$ . This is equivalent to choose the *anchor mapping* of the algebroid  $\mathcal{B}$  to be:

$$\rho^{\mathcal{B}} = (\text{Id}_{n_L} - \Psi^T). \quad (9)$$

The anchor mapping is transferring the elements of  $F$  (that is considered as an abstract vector space itself) to the corresponding admissible velocities (considered as vectors of the vector space  $T_p \mathbb{R}^n$ ). This expression for the anchor mapping is, of course, related by a diffeomorphism with the construction of [2]. If we denote that case by  $\mathcal{A}$ , there exists a mapping  $\Phi_{\mathcal{A}}^{\mathcal{B}}$  which relates (9) with the anchor mapping  $\rho^{\mathcal{A}}$  in the form  $\rho^{\mathcal{B}} = \Phi_{\mathcal{A}}^{\mathcal{B}} \rho^{\mathcal{A}}$ . Consequently, both approaches will be equivalent. We will proceed with the case of  $\mathcal{B}$ .

A Lie algebroid structure is well defined on a bundle as soon as we provide a set of structure constants  $\{c_{\alpha\beta}^{\gamma}\}$  for the Lie algebra and the anchor mapping  $\rho^{\mathcal{B}} : \mathcal{B} \rightarrow T\mathbb{R}^n$ . The anchor mapping is defined by (9) and we fix the structure constants in this basis to zero. In the same way we can obtain the restriction of the original Lagrangian function (1) defined on  $T\mathbb{R}^n$  to this subbundle  $\mathcal{B}$ . The result is trivially found to be, in coordinates  $(q^i, \nu^{\alpha})$  for the points of  $\mathcal{B}$ :

$$L^{\mathcal{B}} = \frac{1}{2} \sum_{\alpha=1}^{n_L} L_{\alpha} (\nu^{\alpha})^2 - \sum_{j=1}^{n_C} \frac{1}{2C_j} q_{C_j}^2. \quad (10)$$

The Euler-Lagrange equations for a Lie algebroid are known from [15]. In the present case, they become (written in coordinates  $\{q^i, \nu^{\alpha}\}$ ):

$$\frac{dq^i}{dt} = \sum_{\alpha} (\rho^{\mathcal{B}})^i_{\alpha} (\nu^{\alpha}); \quad \sum_{\beta=1}^{n_L} L_{\alpha\beta} \dot{\nu}^{\beta} = - \sum_{j=1}^{n_C} (\rho^{\mathcal{B}})^{\alpha}_{j} \frac{q_{C_j}}{C_j}.$$

In our case, they appear to be most suitable for the description of the physically meaningful dynamics (see [2]). In matrix notation we have:

$$\frac{dq_L}{dt} = \nu \quad \frac{dq_C}{dt} = \Psi \nu \quad \frac{d\nu}{dt} = -L^{-1} \Psi^T C^{-1} q_C. \quad (11)$$

This set of equations is quite similar to the system (7). We are going to prove now the relations that exists between them. Let us start with the following lemma.

**Lemma 3.1** *There exists a natural projection  $\Pi : \mathcal{B} \rightarrow T\mathcal{L}$  such that the Lagrangian of the Lie algebroid model is mapped onto the Lagrangian  $L^{T\mathcal{L}}$ .*

**Proof:** The base of the algebroid  $\mathcal{B}$  is, as we have seen above, equal to  $\mathcal{C} \times \mathcal{L}$ , while its fibre has been denoted as  $F$  (in (8)).  $F$  is trivially isomorphic to the space  $T_q\mathcal{L}$  defined by the set of inductance currents  $\{q_{L_1}, \dots, q_{L_{n_L}}\}$ . We can write then that there is a diffeomorphism  $\Psi^B : \mathcal{B} \rightarrow T\mathcal{L} \times \mathcal{C}$ . The composition of this mapping and the natural projection on the first factor  $\pi : T\mathcal{L} \times \mathcal{C} \rightarrow T\mathcal{L}$  defines the desired mapping  $\Pi = \pi \circ \Psi^B : \mathcal{B} \rightarrow T\mathcal{L}$ . On the other hand, charge conservation (5) defines a special subset of the set  $\mathcal{C} \times \mathcal{L}$ , namely the space  $\Psi\mathcal{L} \times \mathcal{L}$  (in geometrical terms this corresponds to the integral submanifold of the foliation defined by  $F$ ; see [9]). Obviously, this space is isomorphic to the space  $\mathcal{L}$ . As the Lagrangian  $L^B$  does not depend on  $q_L$ , it is invariant when restricted to the submanifold  $\tilde{\mathcal{B}}$  of  $\mathcal{B}$  defined as:  $\tilde{\mathcal{B}} = \Pi^*(T\mathcal{L}) = \Psi\mathcal{L} \times \mathcal{L} \times F$ . This set is diffeomorphically equivalent to  $T\mathcal{L}$ . In such a case, the Lagrangian can be pulled back straightforwardly, matching (6). ■

Hence, we have:

**Lemma 3.2** *The solutions of the Euler-Lagrange equations with the Lagrangian  $L^B$  on the Lie algebroid  $\mathcal{B}$  are in one-to-one relation with the solutions of Euler-Lagrange equations with Lagrangian  $L^{T\mathcal{L}}$  on  $T\mathcal{L}$  together with the charge conservation equations.*

**Proof:** The Euler-Lagrange equations to be compared are (7) and (11). It is clear that from the formal point of view both equations are equivalent, if we take into account the charge conservation condition. The diffeomorphism between the fibre of  $\mathcal{B}$  and of  $T\mathcal{L}$  together with the triviality of the bundles imply that the dynamics for the fibre equations are equivalent. The dynamics for  $T\mathcal{L}$  determines, because of last lemma, a curve on  $\tilde{\mathcal{B}}$  which trivially coincides with the solution of the Euler-Lagrange equation corresponding to  $L^B$  whose initial condition coincides with the image  $\Pi^{-1}$  of the initial condition of the curve in  $T\mathcal{L}$  (we are neglecting here the possible ambiguity of the initial charge). ■

Therefore, the algebroid  $\mathcal{B}$  (or equivalently the space  $T\mathcal{L}$ ) is the best geometric option in order to provide the Euler-Lagrange formalism for the LC circuits. The description is physically meaningful, it does not contain constraints and since the Lagrangian is regular, it provides a straightforward Hamiltonian construction. The Hamiltonian construction is also written in terms of the physically meaningful variables, and, as we are going to see in the next section, it coincides with the successful approach as proposed in [7].

#### 4 The Hamiltonian version

The main problem of [11, 12] arises from the singularity of the Lagrangian on  $T\mathbb{R}^n$  which does not allow the use of the Legendre transform and hence a connection with a Hamiltonian description to be compared with others existing in the literature ([7]). As we saw in previous sections, the formulation of the system in terms of Lie algebroids defines a regular Lagrangian. For these cases, we can construct an equivalent Hamiltonian formalism on the dual of the algebroid, because it can be proved (see [15, 2]) that this set is a Poisson manifold, i.e. it admits a Poisson tensor. In coordinates  $(q, \varphi)$  dual to the coordinates  $(q, \nu)$  of  $\mathcal{B}$  the Poisson structure takes the form:

$$\{q^i, q^j\} = 0 \quad \{q^i, \varphi_\alpha\} = \rho_\alpha^i \quad \{\varphi_\alpha, \varphi_\beta\} = c_{\alpha\beta}^\gamma \varphi_\gamma = 0. \quad (12)$$

The choice of  $\mathcal{B}$  allows us to use also the Legendre transform and to formulate a Hamiltonian formalism in a very simple way. The Legendre transform is simply written as the fibre derivative of  $L$ , and in coordinates this means that we define the momenta to be:

$$\varphi_i = \frac{\partial L^B}{\partial \nu^i} = L_i \nu^i. \quad (13)$$

Since the fibre of  $\mathcal{B}$  and the fibre of  $T\mathcal{L}$  are equivalent, this definition implies that the fibre of  $\mathcal{B}^*$  is isomorphic to the vector space  $\mathcal{M}$  defined by the magnetic fluxes of the inductances, which is the fibre of  $T^*\mathcal{L}$ , i.e. the magnetic fluxes (that we can call physical) are the momenta resulting from the action of the Legendre transform on  $L^{T\mathcal{L}}$  as  $\varphi_{L_i} = L_i \dot{q}_{L_i}$ . This expression coincides with the physical definition of magnetic flux on an inductance, and corresponds to the elements of  $\mathcal{F}^*$  defined in [2]. So we work now on the bundle  $\mathcal{B}^*$ , which is also a trivial bundle defined on the base  $\mathbb{R}^n = \mathcal{C} \times \mathcal{L}$  and whose fibre coordinates are those defined by (13). In the coordinates  $\{q_L, q_C, \varphi\}$ , the Hamiltonian becomes:

$$H = \sum_{i=1}^{n_L} \frac{1}{2L_i} \varphi_i^2 + \sum_{i=1}^{n_C} \frac{1}{2C_i} q_{C_i}^2.$$

But, the introduction of the manifold  $T\mathcal{L}$  also yields a regular Lagrangian description. This implies that there is an equivalent Hamiltonian description on  $T^*\mathcal{L}$ , associated to the canonical symplectic structure of the cotangent bundle. Our goal is to relate this Hamiltonian system with the system defined on  $\mathcal{B}^*$  and other solutions as obtained in e.g., [1, 7]. To do this, we need the following lemma.

**Lemma 4.1** *There is a Poisson morphism connecting the Poisson structure associated to the canonical symplectic structure on  $T^*\mathcal{L}$  and the Poisson structure defined in [7].*

**Proof:** The definition of the mapping  $\Psi$  reads  $q_C = \Psi q_L$ . Since the cotangent bundle is in this case a trivial bundle, we can write  $T^*\mathcal{L} = \mathcal{L} \times \mathcal{M}$  where we denote by  $\mathcal{M}$  the space of magnetic fluxes on the inductances  $\{\varphi_{L_i}\}$ . We can trivially extend the mapping  $\Psi$  to this space as  $\Psi : T^*\mathcal{L} \rightarrow \mathcal{C} \times \mathcal{M}$ , and hence the canonical Poisson tensor on  $T^*\mathcal{L}$  which is written in the form

$$P = \sum_{ij} \begin{pmatrix} 0 & \text{Id} \\ -\text{Id} & 0 \end{pmatrix}_{ij} \frac{\partial}{\partial q_{L_i}} \wedge \frac{\partial}{\partial \varphi_{L_i}},$$

is transformed under the linear mapping  $\Psi$  on a tensor on  $\mathcal{C} \times \mathcal{M}$  which in the corresponding coordinates  $\{q_{C_i}, \varphi_{L_j}\}$  will be written as:

$$\Psi_*(P) = \begin{pmatrix} 0 & \Psi \\ -\Psi^T & 0 \end{pmatrix}. \quad (14)$$

This expression coincides with the Poisson tensor proposed in [1, 7], since  $\Psi$  coincides with the matrix  $B_{LC}$  as we saw in previous sections of the paper. This implies that the canonical Hamiltonian formalism on  $T^*\mathcal{L}$  is equivalent to the Hamiltonian formalism defined on  $\mathcal{C} \times \mathcal{M}$  by the Poisson structure (14). ■

We can also relate this Poisson tensor with the canonical Poisson tensor defined on the dual of the Lie algebroid  $\mathcal{B}$ . This Poisson tensor takes the form (12). The corresponding expression is:

$$\begin{pmatrix} 0 & \text{Id}_{n_L} & -\Psi^T \\ -\text{Id}_{n_L} & \Psi & 0 \end{pmatrix}.$$

Furthermore, this Poisson structure can be related with (14) on  $\mathcal{C} \times \mathcal{M}$  (and hence with the Hamiltonian structure on  $T^*\mathcal{L}$ ). Since  $\mathcal{B}^*$  is a trivial bundle  $\mathcal{B}^* = \mathcal{C} \times \mathcal{L} \times \mathcal{B}^*$ , where  $\mathcal{B}^*$  is the fibre, it makes sense to consider the subspace of  $\mathcal{B}^*$  as  $\mathcal{C} \times \mathcal{B}^* \subset \mathcal{B}^*$ . The restriction of the Poisson structure of  $\mathcal{B}^*$  to this subspace gives as a result (14). Since the Hamiltonian on  $\mathcal{B}^*$  is invariant when restricted to  $\mathcal{C} \times \mathcal{B}^*$  it is evident that the Hamiltonian dynamics is equivalent to that of  $\mathcal{B}^*$ . But we know that  $\mathcal{B}^*$  is trivially isomorphic to  $\mathcal{M}$ ; hence the dynamics of the Hamiltonian system defined on  $\mathcal{B}^*$  is equivalent to the systems defined in [1, 7]. Let us next consider a simple example to illustrate the different methods.

## 5 Example

Consider the LC circuit depicted in Figure 1. This example is also used in [1, 6]. We first derive the equations of motion on the space  $T\mathcal{L}$ , which is defined by the coordinates  $\{q_{L_i}, \dot{q}_{L_i}\}$  for  $i = 1, 2, 3, 4$ . The matrix  $\Psi$  is readily found using Kirchhoff's current law  $q_C - \Psi \dot{q}_L = 0$ , i.e.,

$$\Psi = \begin{pmatrix} 1 & 0 & 0 & 1 \\ -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & -1 \end{pmatrix}.$$

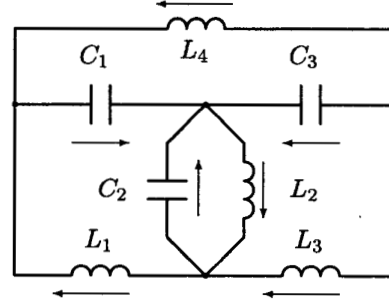


Figure 1: Example LC circuit.

The Lagrangian  $L^{T\mathcal{L}}$  for the circuit is

$$\begin{aligned} L^{T\mathcal{L}} = & \frac{1}{2} \sum_{i=1}^4 L_i \dot{q}_{L_i}^2 - \frac{1}{2C_1} (q_{L_1} + q_{L_4})^2 \\ & - \frac{1}{2C_2} (-q_{L_1} + q_{L_2} + q_{L_3})^2 \\ & - \frac{1}{2C_3} (-q_{L_3} - q_{L_4})^2. \end{aligned} \quad (15)$$

Plugging the latter into the Euler-Lagrange equations (7) yields together with the charge conservation equations:

$$\begin{aligned} q_{C_1} &= q_{L_1} + q_{L_4} & q_{C_2} &= -q_{L_1} + q_{L_2} + q_{L_3} \\ q_{C_3} &= -q_{L_3} - q_{L_4} \end{aligned} \quad (16)$$

the equations of motion for the circuit in Figure 1.

$$\begin{aligned} \ddot{q}_{L_1} &= -\frac{1}{L_1 C_1} (q_{L_1} + q_{L_4}) + \frac{1}{L_1 C_2} (-q_{L_1} + q_{L_2} + q_{L_3}) \\ \ddot{q}_{L_2} &= -\frac{1}{L_2 C_2} (-q_{L_1} + q_{L_2} + q_{L_3}) \\ \ddot{q}_{L_3} &= -\frac{1}{L_3 C_2} (-q_{L_1} + q_{L_2} + q_{L_3}) - \frac{1}{L_3 C_3} (-q_{L_3} - q_{L_4}) \\ \ddot{q}_{L_4} &= -\frac{1}{L_4 C_1} (q_{L_1} + q_{L_4}) - \frac{1}{L_4 C_3} (-q_{L_3} - q_{L_4}), \end{aligned}$$

Next, we consider the dynamics on  $\mathcal{B}$ . In this case, the choice for the coordinates of  $\mathcal{B}$  is  $\{q_{L_i}, q_{C_j}, \nu_i\}$  for  $i = 1, 2, 3, 4$  and  $j = 1, 2, 3$ . The expression for the anchor mapping is just:

$$\rho^{\mathcal{B}} = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 \end{pmatrix}$$

The corresponding Lagrangian turns out to be:

$$\begin{aligned} L^{\mathcal{B}} = & \frac{1}{2} (L_1(\nu^1)^2 + L_2(\nu^2)^2 + L_3(\nu^3)^2 + L_4(\nu^4)^2) - \\ & - \frac{1}{2C_1} q_{C_1}^2 - \frac{1}{2C_2} q_{C_2}^2 - \frac{1}{2C_3} q_{C_3}^2. \end{aligned}$$

Hence, the Euler-Lagrange equations on  $\mathcal{B}$  become:

$$\begin{aligned} \dot{\nu}^1 &= -\frac{1}{L_1 C_1} q_{C_1} + \frac{1}{L_1 C_2} q_{C_2} \\ \dot{\nu}^2 &= -\frac{1}{L_2 C_2} q_{C_2} \\ \dot{\nu}^3 &= -\frac{1}{L_3 C_2} q_{C_2} + \frac{1}{L_3 C_3} q_{C_3} \\ \dot{\nu}^4 &= -\frac{1}{L_4 C_1} q_{C_1} + \frac{1}{L_4 C_3} q_{C_3}, \end{aligned}$$

with  $\frac{dq_{L_i}}{dt} = \nu^i$  and

$$q_{\dot{C}_1} = \nu^1 + \nu^4 \quad q_{\dot{C}_2} = -\nu^1 + \nu^2 + \nu^3 \quad q_{\dot{C}_3} = -\nu^3 - \nu^4.$$

It is easily seen that the latter set of equations coincides with the set defined on  $T\mathcal{L}$ . Let us proceed by equating the Hamiltonian descriptions on  $T^*\mathcal{L}$  and  $\mathcal{B}^*$ . First, the Hamiltonian  $H^{T^*\mathcal{L}}$  is obtained from the Lagrangian (15) by the Legendre transform  $\varphi_{L_i} = L_i \dot{q}_{L_i}$ . The resulting Hamiltonian is:

$$H^{T^*\mathcal{L}} = \sum_{i=1}^4 \frac{1}{2L_i} \varphi_{L_i}^2 + \frac{1}{2C_1} (q_{L_1} + q_{L_4})^2 + \frac{1}{2C_2} (-q_{L_1} + q_{L_2} + q_{L_3})^2 + \frac{1}{2C_3} (-q_{L_3} - q_{L_4})^2.$$

The corresponding Hamilton equations of motion turn out to be  $\frac{dq_{L_i}}{dt} = \frac{\varphi_i}{L_i}$  for  $i = 1, 2, 3, 4$  and

$$\begin{aligned} \dot{\varphi}_1 &= -\frac{1}{C_1} (q_{L_1} + q_{L_4}) + \frac{1}{C_2} (-q_{L_1} + q_{L_2} + q_{L_3}) \\ \dot{\varphi}_2 &= -\frac{1}{C_2} (-q_{L_1} + q_{L_2} + q_{L_3}) \\ \dot{\varphi}_3 &= -\frac{1}{C_2} (-q_{L_1} + q_{L_2} + q_{L_3}) - \frac{1}{C_3} (-q_{L_3} - q_{L_4}) \\ \dot{\varphi}_4 &= -\frac{1}{C_1} (q_{L_1} + q_{L_4}) - \frac{1}{C_3} (-q_{L_3} - q_{L_4}). \end{aligned}$$

We can obtain them also working on  $\mathcal{B}^*$  as follows. The Hamiltonian function regarding the formulation at  $\mathcal{B}^*$  reads:

$$H^{\mathcal{B}^*} = \sum_{i=1}^4 \frac{1}{2L_i} \varphi_i^2 + \sum_{i=1}^3 \frac{1}{2C_i} q_{C_i}^2,$$

where the momenta are just  $\varphi_i = \frac{\partial L^{\mathcal{B}}}{\partial \nu^i} = L_i \nu^i$ . Consequently, by using  $\nu^i = \frac{\varphi_i}{L_i}$  the Hamilton equations of motion become:

$$\begin{aligned} q\dot{C}_1 &= \frac{1}{L_1} \varphi_1 + \frac{1}{L_4} \varphi_4 & q\dot{C}_2 &= -\frac{1}{L_1} \varphi_1 + \frac{1}{L_2} \varphi_2 + \frac{1}{L_3} \varphi_3 \\ q\dot{C}_3 &= -\frac{1}{L_3} \varphi_3 - \frac{1}{L_4} \varphi_4, \end{aligned}$$

and

$$\begin{aligned} \dot{\varphi}_1 &= -\frac{1}{C_1} q_{C_1} + \frac{1}{C_2} q_{C_2} & \dot{\varphi}_2 &= -\frac{1}{C_2} q_{C_2} \\ \dot{\varphi}_3 &= -\frac{1}{C_2} q_{C_2} + \frac{1}{C_3} q_{C_3} & \dot{\varphi}_4 &= -\frac{1}{C_1} q_{C_1} + \frac{1}{C_3} q_{C_3}. \end{aligned}$$

It is again seen that the latter coincides with the solutions on  $T^*\mathcal{L}$  if we keep the equations of conservation of charge (16).

## 6 Conclusions and future research

This paper presents two new choices of regular Lagrangians for the descriptions of non switched LC networks, and proves that the solutions are equivalent one to another and to the results obtained in the literature. The methods turn out to be fairly effective and simple to deal with. They provide equivalent Hamiltonian formalisms, also effective and simple, and it is proved that they are equivalent to other well known approaches.

We intend to continue this line of research and extend these results to the case of switched networks, whose (singular) Lagrangian formalism has been studied in [12, 11].

## Acknowledgement

The authors want to thank Prof. Arjan van der Schaft for the preliminary discussions regarding the connection between the constraint Lagrangian equations and the implicit Hamiltonian framework.

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